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A Projection-Iterative Method for Finding Periodic Solutions of Nonlinear Systems of Difference-Differential Equations with Impulses

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A method is developed for finding approximately the periodic solutions of nonlinear systems of difference-differential equations with impulses at fixed moments.

During recent years which have witnessed vigorous advances in the theory of automatic control, the theory of nonlinear oscillations, quantum mechanics, and so on, a number of contributions have been published concerning solutions of differential equations with impulses. Mil'man and Myshkis were the first to investigate these problems in [1, 2]; notice also [3–11].

The present paper supplies a justification of a method for the study of periodic solutions of nonlinear systems of differential-difference equations with impulses at fixed moments. The method employs and unifies the ideas of the Galerkin method and the numerical-analytic method due to Samoilenko [3].

1. STATEMENT OF THE PROBLEM. SOME ASSUMPTIONS

Consider a system of difference-differential equations with impulses at fixed moments:

$$\begin{aligned} \dot{x} &= f(t, x(t), x(t-h)), \qquad t \neq t_i \\ \Delta x|_{t=t_i} &= I_i(x(t_i)), \end{aligned} \tag{1}$$

where $x, f, I_i \in R^n$, $t_i \in R$ $(i \in Z_0)$ are fixed points, Z_0 is a set of integers, $\Delta x|_{t=t_i} = x(t_i+0) - x(t_i-0)$, and $t_{i+1} > t_i$ for $i \in Z_0$.

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By P denote the point with coordinates $(t, x(t))$, where $x(t)$ is the solution of (1). The motion of the point P can be described as follows: the point P starts at the point (τ_0, x_0) and moves along the integral curve $(t, x(t))$ of the system $\dot{x} = f(t, x(t), x(t-h))$ until the moment $t_1 > \tau_0$ when the point P "instantly" moves from the position $(t_1, x(t_1))$ into the position $(t_1, x(t_1) + I_1(x(t_1)))$. Then the point P moves along the integral curve $(t, x(t))$ of the system without impulses until the moment $t_2 > t_1$, and so on. Without loss of generality it can be assumed that $\tau_0 = 0$.

Therefore, the solution of the system with impulses (1) is a piecewise continuous function $x(t)$ with discontinuity points of the first kind at the points t_i, $i \in Z_0$, and for $t \in (t_{i-1}, t_i)$ it satisfies the equation

$$\dot{x} = f(t, x(t), x(t-h)),$$

while for $t = t_i$, it satisfies the jump condition

$$x(t_i + 0) - x(t_i - 0) = I_i(x(t_i - 0)).$$

Suppose the function $x(t)$ is left continuous at the jump points t_i, $i \in Z_0$, i.e.,

$$x(t_i) = x(t_i - 0) = \lim_{\varepsilon \downarrow 0} x(t_i - \varepsilon).$$

Suppose the points t_i are such that

$$\lim_{i \to \pm\infty} t_i = \pm\infty.$$

We say that conditions (A) hold if the following conditions are satisfied:

A1. The function $f: G \to R^n$ is defined and continuous in

$$G = \{(t, x, y): t \in R, x, y \in \bar{D}\}$$

it is periodic with respect to t with period 2π and satisfies the inequality

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leqslant N\{|x - \bar{x}| + |y - \bar{y}|\}, x, \bar{x}, y, \bar{y} \in \bar{D}, \tag{2}$$

where $\bar{D}$ is the closure of a bounded domain in R^n, $|\cdot|$ is some norm in R^n.

A2. Natural numbers κ and κ_1 exist, such that $t_{i+\kappa} = t_i + h$ and $\kappa_1 h = 2\pi$.

A3. The functions $I_i: \bar{D} \to R^n$ $(i \in Z_0)$ are defined and continuous in $\bar{D}$ and satsisfy the conditions

$$(|I_i(x) - I_i(\bar{x})| \leqslant L\,|x - \bar{x}|, \qquad x, \bar{x} \in \bar{D} \tag{3}$$

uniformly in i,

$$I_{i+\rho}(x) = I_i(x), \qquad x \in \bar{D}$$

where $\rho = \kappa\kappa_1$.

Introduce the notation

$$M = \max_{\substack{t \in [0, 2\pi] \\ x, y \in \bar{D}}} |f(t, x, y)| + \max_{\substack{x \in \bar{D} \\ 1 \leqslant i \leqslant \rho}} |I_i(x)|.$$

Consider the space Ω_1 consisting of all continuous 2π-periodic functions. Each function $v(t) \in \Omega_1$ is associated with the corresponding Fourier series

$$v(t) \sim \frac{a_0}{2} + \sum_{q=1}^{\infty} (a_q \cos qt + b_q \sin qt),$$

where

$$a_q = \frac{1}{\pi} \int_0^{2\pi} v(t) \cos qt \, dt, \qquad b_q = \frac{1}{\pi} \int_0^{2\pi} v(t) \sin qt \, dt.$$

Introduce the operators $P_0 \colon \Omega_1 \to R^n$ and $P_m \colon \Omega_1 \to R^n$ defined by the equalities

$$P_0 v(t) = \frac{1}{2\pi} \int_0^{2\pi} v(t) \, dt,$$

$$P_m v(t) = \sum_{q=1}^{m} (a_q \cos qt + b_q \sin qt).$$

Introduce the notation

$$\|x(t)\| = \sup_{t \in [0, 2\pi]} |x(t)|.$$

Further the following lemma will be employed.

LEMMA 1 [12]. *If $v(t)$ is a continuous 2π-periodic function, then the following estimates hold*:

$$\left\| \int_0^t P_m v(t) \, dt \right\| \leqslant \pi \, \|v(t)\|, \tag{4}$$

$$\left\| \int_0^t (E - P_0 - P_m) \, v(t) \, dt \right\| \leqslant 2\sqrt{2} \, \sigma(m) \, \|v(t)\|, \tag{5}$$

where E is the identity,

$$\sigma(m)=\begin{cases}\dfrac{\pi^2}{6}, & m=0\\[2ex] \dfrac{1}{(m+1)^2}+\dfrac{1}{(m+2)^2}+\cdots, & m=1,2,\ldots.\end{cases}$$

Note that inequalities (4) and (5) also hold for piecewise continuous functions admitting Fourier expansions.

The periodic solutions of system (1) are found in two steps. Assume first that a unique 2π-periodic solution of system (1) exists and construct a sequence of functions, periodic and tending uniformly to the solution of system (1). Then we prove an existence theorem for the unique periodic solution of system (1).

2. Constructing Successive Approximations. Auxiliary Theorems

Suppose system (1) has a unique 2π-periodic solution $\varphi(t)$ for which $\varphi(0)=x_0$.

Consider the set S consisting of all piecewise continuous functions $x\colon [-h, 2\pi]\to \bar{D}$ with discontinuity points t_i, $i\in Z_0$ of the first kind, for which

$$x(0)=x(2\pi)=x_0, \tag{6}$$

$$x(t)=x(t+2\pi) \qquad \text{for} \quad t\in[-h,0]. \tag{7}$$

Define the sequence of functions $x_\kappa(t)$ by the formula, as follows:

$$x_0(t)\equiv x_0 \qquad \text{for} \quad t\in[-h,2\pi],$$

$$x_{\kappa+1}(t)=\begin{cases} x_0+\displaystyle\int_0^t P_m f(t, x_{\kappa+1}(t), x_{\kappa+1}(t-h))\,dt \\[2ex] \quad+\displaystyle\int_0^t (E-P_0-P_m)\, f(t, x_\kappa(t), x_\kappa(t-h))\,dt \\[2ex] \quad+\displaystyle\sum_{0<t_i<t} I_i(x_\kappa(t_i))-\frac{t}{2\pi}\sum_{0<t_i<2\pi} I_i(x_\kappa(t_i)) \qquad \text{for} \quad t\in[0,2\pi] \\[2ex] x_{\kappa+1}(t+2\pi) \qquad \text{for} \quad t\in[-h,0), \end{cases} \tag{8}$$

where $m>0$ is a fixed integer.

By Ω denote the set of points from R^n lying in the domain D together with their $a(m)$-neighbourhood, where $a(m)=M(\pi+2\sqrt{2}\,\sigma(m)+2\rho)$.

We say that conditions (B) hold if the following conditions are satisfied:

B1. The set Ω is non-empty.

B2. The relation $x_0 \in \Omega$ holds.

B3. The following inequalities hold:

$$(4/\sqrt{3})\,\pi N < 1, \qquad \frac{4\sqrt{2}\,\sigma(m)N + 2\rho L}{1-(4/\sqrt{3})\,\pi N} < 1.$$

The algorithm for finding the functions $x_{\kappa+1}(t)$, $\kappa = 0, 1, 2, \ldots$ is now given.

In Eq. (8) set $\kappa = 0$ and for $t \in [0, 2\pi]$ we obtain

$$x_1(t) = x_0 + \int_0^t P_m f(t, x_1(t), x_1(t-h))\,dt$$
$$+ \int_0^t (E - P_0 - P_m)\, f(t, x_0, x_0)\,dt$$
$$+ \sum_{0<t_i<t} I_i(x_0) - \frac{t}{2\pi} \sum_{0<t_i<2\pi} I_i(x_0).$$

Introduce the notation

$$a_{q1} = \frac{1}{\pi} \int_0^{2\pi} f(t, x_1(t), x_1(t-h)) \cos qt\,dt, \tag{9}$$

$$b_{q1} = \frac{1}{\pi} \int_0^{2\pi} f(t, x_1(t), x_1(t-h)) \sin qt\,dt, \tag{10}$$

$$\psi_0(t) = \int_0^t (E - P_0 - P_m)\, f(t, x_0, x_0)\,dt$$
$$+ \sum_{0<t_i<t} I_i(x_0) - \frac{t}{2\pi} \sum_{0<t_i<2\pi} I_i(x_0).$$

Then the solution $x_1(t)$ can be rewritten as

$$x_1(t) = x_0 + \sum_{q=1}^{m} \frac{1}{q}\,[a_{q1} \sin qt + b_{q1}(1 - \cos qt)] + \psi_0(t). \tag{11}$$

Relation (11) implies that the solution $x_1(t)$ depends on $2mn$ unknown numbers a_{q1} and b_{q1} $(q = \overline{1, m})$. These parameters can be determined by the system (9), (10) from $2mn$ algebraic or transcendent equations with $2mn$

unknown parametrs (for fixed m). Analogously, (for fixed m) the $(\kappa+1)$th approximation $x_{\kappa+1}(t)$ can be determined by the equality

$$x_{\kappa+1}(t)=x_0+\sum_{q=1}^{m}\frac{1}{q}\left[a_{q,\kappa+1}\sin qt+b_{q,\kappa+1}(1-\cos qt)\right]+\psi_{\kappa+1}(t), \quad (12)$$

where

$$\psi_{\kappa+1}(t)=\int_0^t (E-P_0-P_m)\, f(t, x_\kappa(t), x_\kappa(t-h))\, dt$$
$$+\sum_{0<t_i<t} I_i(x_\kappa(t_i))-\frac{t}{2\pi}\sum_{0<t_i<2\pi} I_i(x_\kappa(t_i)), \quad (13)$$

and the coefficients $a_{q,\kappa+1}$ and $b_{q,\kappa+1}$ are determined by the system

$$\begin{aligned} a_{q,\kappa+1}&=\frac{1}{\pi}\int_0^{2\pi} f(t, x_{\kappa+1}(t), x_{\kappa+1}(t-h))\cos qt\, dt,\\ b_{q,\kappa+1}&=\frac{1}{\pi}\int_0^{2\pi} f(t, x_{\kappa+1}(t), x_{\kappa+1}(t-h))\sin qt\, dt. \end{aligned} \quad (14)$$

LEMMA 2. *Suppose conditions* (A) *and* (B) *hold. Then equations* (8) *can be solved with respect to* $x_\kappa(t)$ *and* $x_\kappa(t)\in S$, $\kappa=1, 2,\ldots$.

Proof. Define the operator $T\colon S\to R^n$ by the formula

$$T(x, Z)(t)=\begin{cases} x_0+\displaystyle\int_0^t P_m f(t, x(t), x(t-h))\, dt \\ \quad +\displaystyle\int_0^t (E-P_0-P_m)\, f(t, Z(t), Z(t-h))\, dt \\ \quad +\displaystyle\sum_{0<t_i<t} I_i(Z(t_i))-\frac{t}{2\pi}\sum_{0<t_i<2\pi} I_i(Z(t_i)), & t\in[0, 2\pi] \\ T(x, Z)(t+2\pi), & t\in[-h, 0). \end{cases}$$

Then for $x\in S$ and a fixed $Z\in S$, the equality

$$T(x, Z)(0)=T(x, Z)(2\pi)=x_0$$

holds.

Inequalities (4) and (5) yield the estimate

$$\|T(x, Z)(t)-x_0\|\leqslant a(m).$$

Hence the operator T transforms the set S into itself (for fixed $Z \in S$).

Let $x(t)$, $\bar{x}(t) \in S$. Estimate the norm of the difference $T(x, Z)(t) - T(\bar{x}, Z)(t)$,

$$\|T(x, Z)(t) - T(\bar{x}, Z)(t)\| \leqslant (4/\sqrt{3})\, \pi N \,\|x(t) - \bar{x}(t)\|. \tag{15}$$

Condition B3 and inequality (15) imply that the operator $T: S \to S$ is contractive and by the Banach fixed point theorem the operator equation (8) has a unique solution $x_{\kappa+1}(t) \in S$, $\kappa \geqslant 0$.

Moreover, the following relations hold:

$$\begin{aligned} &\|x_{\kappa+1}(t) - x_0\| \leqslant a(m) \\ &x_{\kappa+1}(0) = x_{\kappa+1}(2\pi) = x_0. \end{aligned} \tag{16}$$

This proves Lemma 2.

3. Convergence of the Successive Approximations

LEMMA 3. *Suppose conditions* (A) *and* (B) *hold. Then the sequence of functions* $\{x_\kappa(t)\}_0^\infty$ *defined by Eqs.* (8) *is uniformly convergent as* $t \in [-h, 2\pi]$.

Proof. Inequalities (2), (3), (4), and (5) imply the estimate

$$\begin{aligned} &\|x_2(t) - x_1(t)\| \\ &\quad \leqslant \pi N \,\|x_2(t) - x_1(t)\| + \pi N \,\|x_2(t-h) - x_1(t-h)\| \\ &\qquad + 2\sqrt{2}\, \sigma(m)\, N[\|x_1(t) - x_0\| + \|x_1(t-h) - x_0\|] \\ &\qquad + 2\rho L \,\|x_1(t) - x_0\| \\ &\quad \leqslant (4/\sqrt{3})\, \pi N \,\|x_2(t) - x_1(t)\| + (4\sqrt{2}\, \sigma(m)\, N + 2\rho L)\, \|x_1(t) - x_0\|. \end{aligned} \tag{17}$$

Inequality (17) yields

$$\|x_2(t) - x_1(t)\| \leqslant \frac{4\sqrt{2}\, \sigma(m) N + 2\rho L}{1 - (4/\sqrt{3})\, \pi N}\, \|x_1(t) - x_0\|.$$

Introduce the notation

$$q_m = \frac{4\sqrt{2}\, \sigma(m) N + 2\rho L}{1 - (4/\sqrt{3})\, \pi N}. \tag{18}$$

The method of mathematical induction implies

$$\|x_{\kappa+1}(t)-x_\kappa(t)\| \leqslant q_m^\kappa \, \|x_1(t)-x_0\|.$$

Hence, the following inequality holds:

$$\|x_{\kappa+s}(t)-x_\kappa(t)\| \leqslant q_m^\kappa \sum_{i=0}^{s-1} q_m^i \, \|x_1(t)-x_0\|. \tag{19}$$

Inequality (19), in view of condition B3 and the fact that S is closed, implies the uniform convergence of the sequence $\{x_\kappa(t)\}_0^\infty$. Introduce the notation $x_\infty(t)=\lim_{\kappa\to\infty} x_\kappa(t)$. Since the set S is closed, $x_\infty(t)\in S$. As $s\to\infty$, inequality (19) implies

$$\|x_\infty(t)-x_\kappa(t)\| \leqslant \frac{q_m^\kappa}{1-q_m} \|x_1(t)-x_0\|.$$

By the definition of the functions $x_\kappa(t)$, it follows that the function $x_\infty(t)$ for $t\in[0, 2\pi]$ satisfies the equation

$$\begin{aligned} x(t) = x_0 &+ \int_0^t f(t, x(t), x(t-h))\, dt \\ &- \int_0^t P_0 f(t, x(t), x(t-h))\, dt \\ &+ \sum_{0<t_i<t} I_i(x(t_i)) - \frac{t}{2\pi} \sum_{0<t_i<2\pi} I_i(x(t_i)). \end{aligned} \tag{20}$$

The relation $x_\infty(t)\in S$ implies that the function $x_\infty(t)$ has a 2π-periodic extension denoted by $X_\infty(t)$.

THEOREM 1. *Suppose the following conditions hold*:

1. *Conditions* (A) *and* (B) *are satisfied.*
2. *Equation* (1) *has a* 2π*-periodic solution* $\varphi(t)$ *for which* $\varphi(0)=x_0$.

Then the relation

$$\varphi(t)=X_\infty(t)$$

holds uniformly in x_0, *and*

$$\|\varphi(t)-x_\kappa(t)\| \leqslant \frac{q_m^\kappa}{1-q_m} \|x_1(t)-x_0\|,$$

where q_m *are defined by Eqs.* (18).

Proof. Condition 2 of the theorem implies that

$$\int_0^{2\pi} f(t, \varphi(t), \varphi(t-h))\, dt + \sum_{0 < t_i < 2\pi} I_i(\varphi(t_i)) = 0.$$

Hence for $t \in [0, 2\pi]$ the function $\varphi(t)$ satisfies Eq. (20) and the uniqueness of the solution of Eq. (20) implies that

$$\varphi(t) \equiv x_\infty(t) \qquad \text{for} \quad t \in [0, 2\pi]$$

and hence

$$\varphi(t) = X_\infty(t) \qquad \text{for} \quad t \in (-\infty, \infty).$$

4. Existence of a Periodic Solution

The existence of a 2π-periodic solution of the system (1) is considered here. Introduce the notation

$$\Delta(x_0) = P_0 f(t, x_\infty(t), x_\infty(t-h)) + \frac{1}{2\pi} \sum_{0 < t_i < 2\pi} I_i(x_\infty(t_i)),$$

$$\Delta_m(x_0) = P_0 f(t, x_m(t), x_m(t-h)) + \frac{1}{2\pi} \sum_{0 < t_i < 2\pi} I_i(x_m(t_i)).$$

Since the function $x_\infty(t)$ is a solution of Eq. (20) then for $\Delta(x_0) = 0$ the function $x_\infty(t)$ will also satisfy system (1).

Thus, the problem for the existence of a periodic solution of Eq. (1) is related to the problem for the existence of zeros of the function $\Delta(x_0)$. The points x_0 for which $\Delta(x_0) = 0$ are singular points of the mapping $\Delta: \bar{D} \to R^n$. However, since the functions $x_\kappa(t)$, $\kappa \geqslant 1$ are the only ones known, then, as to employ efficiently the method proposed, we have to transform the problem for finding the zeros of the function $\Delta(x_0)$ into a problem for finding the zeros of the function $\Delta_\kappa(x_0)$, $\kappa \geqslant 1$.

Theorem 2. *Suppose the following conditions hold*:

1. *Conditions* (A) *and* (B) *are satisfied.*

2. *A convex closed domain $D_1 \subset \bar{D}$ exists such that for some value of $\kappa \geqslant 1$ the mapping $\Delta_\kappa: \bar{D} \to R^n$ has a unique singular point of nonzero index in D_1.*

3. *On the boundary Γ_{D_1} of the domain D_1 the following inequality holds*

$$\inf_{x \in \Gamma_{D_1}} |\Delta_\kappa(x)| > \frac{q_m^\kappa}{1-q_m} (N+L) \|x_1 - x_0\|.$$

Then system (1) *has a 2π-periodic solution $x^*(t)$ for which $x^*(0) = x_0 \in \bar{D}$.*

Proof. The proof is analogous to that of Theorem 1 from [3], taking into account the inequality

$$|\Delta(x_0) - \Delta_\kappa(x_0)| \leqslant (N+L) \frac{q_m^\kappa}{1-q_m} \|x_1 - x_0\|.$$

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